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Multiconstrained Variational Problems in Magnetohydrodynamics: Equilibrium and Slow Evolution*

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A computational method is proposed for solving magnetohydrodynamical equilibrium problems with prescribed flux and mass within the magnetic surfaces that foliate the plasma. Such problems arise in tokamak modeling, for instance, where they determine either equilibria with given adiabatic profiles or slowly evolving quasi-equilibria governed by the Grad–Hogan equations. The classical variational principles of Kruskal and Kulsrud and Woltjer, which express these problems in terms of energy minimization subject to infinite families of nonlinear, nonlocal constraints, are taken as the basis for a direct method of solution. A natural discretization of the classical constraint families is devised, and an iterative algorithm is developed to solve the resulting optimization problems. A convergence theory for the algorithm is established, and an effective numerical implementation of the method is presented for flux-conserving tokamak equilibria. Some computed examples involving plasma heating and adiabatic compression are described.

1. INTRODUCTION

At present the tokamak concept of magnetic confinement is widely believed to offer the most promising prospect for controlled thermonuclear fusion. In a tokamak, axisymmetric plasma configurations are stably confined in a toroidal vacuum chamber by means of a strong external magnetic field and an induced plasma current. These configurations are not in ideal equilibrium during actual physical experiments, however, because of the presence of heating sources, the variation of applied fields, and the effects of resistivity. Instead, they slowly (or adiabatically) evolve in time as quasi-equilibrium processes. The macroscopic properties of such configurations are described by magnetohydrodynamics (MHD), which models the plasma as a single, conducting fluid surrounded by a vacuum, self-consistently interacting with the external field. Nevertheless, the direct analysis and computation of these processes based on the full system of MHD equations is not feasible, since the time scales for different MHD phenomena range over several orders of magnitude [16]. For this reason, a reduced system of governing equations has been introduced by Grad and Hogan [12], which, by neglecting the inertial terms in the force balance equation, describes the plasma evolution as a sequence of equilibrium configurations with slowly varying parameters. The solution of these slow (or adiabatic) evolution problems in MHD therefore constitutes an important aspect of the modeling of plasma behavior in a magnetic confinement device such as a tokamak.

A procedure for solving the reduced system of equations describing slow evolution has been proposed by Grad and his collaborators [10–13] and has subsequently been developed by others [2–4, 17–23, 26]. The basis of this approach is the so-called "generalized differential equation" (GDE), which arises naturally when magnetic surface averages of the instantaneous equilibrium equations are invoked to retrieve the parameters that determine the equilibrium (the Grad–Shafranov profiles for poloidal current and plasma pressure) from the parameters that determine the evolution (the toroidal flux and mass per unit poloidal flux). This approach furnishes the standard formulation of the so-called flux-conserving equilibrium problem for a tokamak. The unusual structure of the resulting GDE, however, makes it hard to treat analytically and numerically. Consequently, an adequate mathematical justification of the GDE theory has not been given, and a sufficiently flexible and reliable computational scheme based on the GDE formalism is not available.

In the present paper we give an alternative theory of the slow evolution of axisymmetric plasma–vacuum systems, in which the instantaneous equilibrium conditions are formulated variationally. The framework for our theory is

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supplied by the classical variational principle of Kruskal and Kulsrud [20], which is also the foundation of the important computational method of Bauer, Bentancourt, and Garabedian [1]. This principle characterizes equilibrium configurations as minimizers of total energy subject to two *infinite* families of constraints that determine the flux and mass within the magnetic surfaces in the plasma. (More correctly, the "mass" also involves the plasma entropy, and so it is actually a measure of the plasma pressure.) These constraints, being derived from the conserved quantities associated with ideal MHD evolution, provide the natural parametrization of the instantaneous equilibrium solutions occurring in slow evolution problems. For a toroidal device such as a tokamak, whose equilibrium configurations are assumed to be symmetric with respect to the toroidal angle, a simpler variational principle due to Woltjer [27, 28] applies. Under axisymmetry, the magnetic surfaces can be identified with the level surfaces of a poloidal flux function, and the constraints can be expressed conveniently as definite integrals. It is then possible to relax the two *infinite* families of constraints into two corresponding *finite* families, which essentially impose conservation of flux and mass in a volume-averaged sense. With such a discretization/interpolation of the classical constraints, the instantaneous equilibrium problem is reduced to a constrained minimization problem of a standard form, whose equilibrium equations are obtained by the usual Lagrange multiplier rule. These equations are equivalent to a Grad–Shafranov equation [7, 24] (for the poloidal flux function) with profile functions (for the poloidal current and plasma pressure) that are determined by the multipliers in response to the constraints. This direct treatment of the constraints is fundamental to our formulation and distinguishes it from the standard variational formulation of the flux-conserving equilibrium problem [1], which imposes the classical constraints by means of a flux-coordinate representation of the magnetic configurations.

Our multiconstrained variational formulation of the equilibrium problem naturally suggests a numerical method based on the principles of optimization theory. Indeed, the main goal of this paper is to develop an iterative algorithm for solving the equilibrium problem that exploits its special variational structure. The algorithm that we give is simple and robust and enables us to compute equilibrium configurations with low or high β, arbitrary external coil geometry, and general magnetic surface topology in a unified fashion. The present method is an extension of the formulation and algorithm that we developed earlier [6] for a mathematical prototype problem modeling two-dimensional, incompressible plasma configurations.

In Section 2 we consider the full system of ideal MHD equations with distributed sources which, in appropriate dimensionless units, are small and vary slowly in time. By a routine expansion of the governing equations in terms of an appropriate small parameter, we obtain the reduced system of Grad–Hogan equations. We then express the axisymmetric equations of slow (or adiabatic) evolution in terms of scalar unknowns, getting conservation laws for flux and mass, along with the Grad–Shafranov force balance equation. We also display the families of constraints associated with each magnetic surface in the form needed for our subsequent analysis.

In Section 3 we construct the multiconstrained variational problem (P_n). Simply put, (P_n) is the relaxed problem that results from averaging the constraints in the classical variational principle between a family of n magnetic surfaces. By also imposing a constraint on the total (plasma plus vacuum) flux, (P_n) includes the plasma–vacuum free-boundary in a natural manner. A solution of (P_n) therefore satisfies both the equilibrium equations and the plasma–vacuum interface conditions *exactly*; its multipliers define the associated Grad–Shafranov profile functions in terms of a corresponding family of n basis functions of finite-element type. We interpret a quasi-equilibrium solution as a solution of (P_n) whose constraint values evolve in slow time.

We develop our iterative algorithm for solving (P_n) in Sections 4 and 5. The construction of this algorithm, which applies to a class of constrained minimization problems, is based on a technique of linearization and convexification of the given nonlinear and nonconvex constraints at each iteration. Similar although not identical constructions are used in finite-dimensional optimization [9, 25]. For algorithms of this kind, convexity properties guarantee the convergence of the iterates from *any* initial guess. Moreover, the structure of the iterative step implies that the iterates can be generated by solving standard linear subproblems. These attributes of the algorithm guarantee that it is both robust and efficient.

We devote Section 6 to a discussion of some computed results. Since our goal is not to simulate actual tokamak experiments, but rather to demonstrate the computational features of our general method, we choose some simple, generic cases as illustrations of its numerical implementation. Specifically, we consider (1) flux-conserving equilibria with increasing β due to plasma heating, and (2) adiabatically compressed equilibria in an external field produced by increasing currents in the toroidal coils. The same method applies to a great variety of quasi-equilibrium processes for plasma-vacuum systems, and is computationally effective over a wide range of conditions.

For the sake of clarity in the exposition, we ignore plasma resistivity throughout our analysis. It is possible to modify the formulation and algorithm that we describe here in the case of adiabatic evolution due to small sources and slowly varying external fields to treat the case of diffusive evolution due to a small resistivity. However, since different physical issues arise in the latter case, we choose to omit any discussion of resistive effects in this paper.

2. SLOW EVOLUTION EQUATIONS

We begin by recalling the full set of equations governing a plasma–vacuum system confined in a toroidal device such as a tokamak. Under the usual assumptions of ideal magnetohydrodynamics, the equations valid in the plasma region are

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = Q_M \tag{2.1}$$

$$\frac{D\zeta}{Dt} + \zeta \nabla \cdot \mathbf{V} = Q_S \tag{2.2}$$

$$\rho \frac{D\mathbf{V}}{Dt} + \nabla p - \mathbf{J} \times \mathbf{B} = \mathbf{F} \tag{2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{B}) = 0 \tag{2.4}$$

$$\nabla \times \mathbf{B} = \mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0; \tag{2.5}$$

and the equations valid in the vacuum region are

$$\nabla \times \mathbf{B} = \bar{\mathbf{J}}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{2.6}$$

Here, ρ, ζ, p, $\mathbf{V}$, $\mathbf{B}$, and $\mathbf{J}$ denote the mass density, entropy density (per unit volume), pressure, velocity, magnetic field, and current density, respectively. The notation $D/Dt := \partial/\partial t + \mathbf{V} \cdot \nabla$ is used for the convective derivative. We suppose that in the plasma there are distributed sources of mass, entropy, and momentum production given by Q_M, Q_S, and $\mathbf{F}$, respectively, and that in the vacuum there are external field coils carrying a given current density $\bar{\mathbf{J}}$. We assume here and throughout that all equations are expressed in nondimensional variables. The physical quantities representing magnetic field, current density, and plasma pressure are retrieved as $B_0\mathbf{B}$, $(\mu_0 L_0)^{-1} B_0 \mathbf{J}$, $\mu_0^{-1} B_0^2 p$, respectively, in terms of a characteristic length L_0, a characteristic field strength B_0, and a magnetic permeability μ_0.

In constrast to usual practice we use the entropy per unit volume ζ rather than the entropy per unit mass $s = \zeta/\rho$. In this notation the equation of state for an ideal gas is

$$p = e^{\zeta/\rho} \rho^{\gamma}, \tag{2.7}$$

and the internal energy density is the convex function $U(\rho, \zeta) = e^{\zeta/\rho}\rho^{\gamma}/(\gamma - 1)$, where γ is the adiabatic index (ratio of specific heats). All of our subsequent development actually applies to the general case in which $U(\rho, \zeta)$ is taken to be any smooth and strictly convex function, and the equation of state is derived from the thermodynamic relation $p = \rho\, \partial U/\partial \rho + \zeta\, \partial U/\partial \zeta - U$. However, we shall restrict our discussion to the familiar case (2.7) for the sake of definiteness.

The above equations hold in a toroidal region D. On the fixed boundary ∂D, which is assumed to be a perfectly conducting shell, the normal component of $\mathbf{B}$ vanishes. On the free-boundary surface $\mathscr{S}$, the plasma–vacuum interface conditions hold; namely,

$$p = 0, \ \mathbf{n} \cdot \mathbf{B} = 0 \text{ on } \mathscr{S} \text{ and } \mathbf{n} \times \mathbf{B} \text{ is continuous across } \mathscr{S}, \tag{2.8}$$

where $\mathbf{n}$ is the unit normal on $\mathscr{S}$. The fixed and free boundaries are therefore magnetic surfaces at every instant of time and there are no surface currents on the free boundary.

Let us suppose for the moment that the external sources of mass, entropy, and momentum do not exist and that the external current density does not vary in time. Then the above evolutionary problem has a class of static ($\mathbf{V} = 0$) equilibrium ($\partial/\partial t = 0$) solutions. In the plasma region these solutions satisfy the standard equilibrium equations

$$\mathbf{J} \times \mathbf{B} = \nabla p, \qquad \nabla \times \mathbf{B} = \mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0, \tag{2.9}$$

while in the vacuum region they satisfy (2.6); on the fixed and free boundaries they fulfill the conditions stated above. In equilibrium the density ρ and entropy ζ do not have a precise meaning, since only the pressure p, which is the combination (2.7), enters in the governing equations (2.9) and hence can be determined.

Now let us suppose that the external sources are small and vary slowly in time, and that the external currents vary slowly in time. Then it is possible to derive approximate equations describing the slow evolution of a plasma–vacuum system that is almost in equilibrium at every instant of time. In order to obtain these equations we introduce a dimensionless parameter $\varepsilon \ll 1$ and scale the unknowns as

$$\rho = \hat{\rho}, \quad \zeta = \hat{\zeta}, \quad p = \hat{p}, \quad \mathbf{V} = \varepsilon \hat{\mathbf{V}}, \quad \mathbf{B} = \hat{\mathbf{B}},$$

where the caretted unknowns depend on the scaled variables $\hat{x} = x$, $\hat{t} = \varepsilon t$. Such a scaling is based on the assumption that the given external sources and currents can be expressed in the form

$$Q_M = \varepsilon \hat{Q}_M(x, \varepsilon t), \qquad Q_S = \varepsilon \hat{Q}_S(x, \varepsilon t),$$
$$\mathbf{F} = \varepsilon^2 \hat{\mathbf{F}}(x, \varepsilon t), \qquad \mathbf{J} = \hat{\mathbf{J}}(x, \varepsilon t).$$

Upon substituting these expressions into the governing equations (2.1)–(2.6) and the associated fixed and free boundary conditions, and after dropping the carets, we find that all of those equations remain unchanged except for the force balance equation which assumes the form

$$\varepsilon^2 \rho \frac{D\mathbf{V}}{Dt} + \nabla p - \mathbf{J} \times \mathbf{B} = \varepsilon^2 \mathbf{F}. \tag{2.10}$$

Now formally neglecting the $O(\varepsilon^2)$ terms in (2.10), we therefore conclude that the (adiabatically) slow evolution of the plasma–vacuum system caused by the presence of external sources and currents is governed by a system of reduced equations: the conservation laws (2.1) and (2.2), the equilibrium equations (2.9) in the plasma and (2.6) in the vacuum, and the (flux-freezing) induction equation (2.4). Moreover, since only the pressure p enters into the equilibrium force balance equation, this system can be further reduced by amalgamating the two conservation laws for ρ and ζ into

$$\frac{Dp^{1/\gamma}}{Dt} + p^{1/\gamma}\nabla \cdot \mathbf{V} = Q, \tag{2.11}$$

where Q denotes a heating source density. Thus, we have derived the nondissipative Grad–Hogan equations [12].

Henceforth we assume that the plasma–vacuum system is axisymmetric. The toroidal region $D = \{x = (r, \phi, z): (r, z) \in \Omega, 0 \leqslant \phi < 2\pi\}$ is then defined by its cross section Ω in the usual cylindrical coordinates. The invariance of the system with respect to the toroidal angle ϕ implies that the slow evolution equations can be replaced by a simpler system of equations. For the sake of simplicity, we shall assume that the velocity field is purely poloidal, $\mathbf{V} = (V_r, 0, V_z)$, this assumption being consistent with the absence of the inertial term DV_ϕ/Dt in the reduced equations. (It is straightforward to include toroidal flow in what follows without any conceptual changes, because the toroidal momentum behaves exactly like the toroidal magnetic flux.) The axisymmetric slow evolution equations govern the unknowns $\Psi(r, z, t)$, $f(r, z, t)$, and $p(r, z, t)$, where the magnetic field and current density are written in terms of flux functions Ψ and f according to

$$\mathbf{B} = \nabla\Psi \times \nabla\phi + f\nabla\phi \tag{2.12}$$

$$\mathbf{J} = \nabla f \times \nabla\phi + (L\Psi)\nabla\phi, \tag{2.13}$$

with

$$L := -r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) - \frac{\partial^2}{\partial z^2}.$$

In the plasma region these equations are

$$\frac{Dp^{1/\gamma}}{Dt} + p^{1/\gamma}\nabla \cdot \mathbf{V} = Q \tag{2.14}$$

$$-r^{-2}f\nabla f + r^{-2}(L\Psi)\nabla\Psi = \nabla p, \qquad \nabla f \times \nabla\Psi = 0 \tag{2.15}$$

$$\frac{D\Psi}{Dt} = 0 \tag{2.16}$$

$$\frac{D}{Dt}(r^{-2}f) + r^{-2}f\nabla \cdot \mathbf{V} = 0; \tag{2.17}$$

and in the vacuum region they are

$$\nabla f = 0, \qquad L\Psi = r\bar{J}_\phi. \tag{2.18}$$

Here, the vector equations are separated into their poloidal parts (perpendicular to $\nabla\phi$) and their toroidal parts (parallel to $\nabla\phi$). The equilibrium force balance equations (2.15) reduce to the familiar Grad–Shafranov equation

$$\begin{aligned} L\Psi &= f(\Psi, t)f'(\Psi, t) + r^2 p'(\Psi, t), \\ f &= f(\Psi, t), \qquad p = p(\Psi, t), \end{aligned} \tag{2.19}$$

where prime denotes differentiation with respect to Ψ. The unknown profile functions $f(\Psi, t)$ and $p(\Psi, t)$ are called "surface quantities," being constant on the magnetic surfaces $\{\Psi = \sigma\}$.

By virtue of (2.16), each magnetic surface $\{\Psi = \sigma\}$ moves with the plasma flow $\mathbf{V}$ so that the poloidal flux σ is conserved. Hence, under axisymmetry the fixed and free boundary conditions can be normalized as follows. The boundary condition on the shell is taken to be

$$\Psi = 0 \qquad \text{on } \partial\Omega, \tag{2.20}$$

while the plasma–vacuum interface conditions are expressed as

$$p = 0,\ \Psi = \sigma_0 \text{ on } \mathscr{S},\ \nabla\Psi \text{ and } f \text{ are continuous across } \mathscr{S}, \tag{2.21}$$

for a flux constant $\sigma_0 > 0$ that determines the free-boundary surface $\mathscr{S}$. If we make the sign convention that the total plasma current is positive, then we can characterize the plasma region as the set $\{\Psi > \sigma_0\}$. (It is possible to consider more physically realistic boundary conditions and magnetic geometries within the conceptual framework that is developed here. Nevertheless, these complications will be avoided in the present exposition.)

The prescribed external current density $\bar{\mathbf{J}} = \bar{J}_\phi\nabla\phi$ is assumed to be purely toroidal; for definiteness, it may be realized as a finite collection of elementary coils, $\bar{J}_\phi = \Sigma I_l \delta(r - R_l)\,\delta(z - Z_l)$ located at (R_l, Z_l) carrying currents I_l. As usual, poloidal current coils are assumed to be located outside the perfectly conducting shell (see [7, 24]).

The general constraints of motion governed by axisymmetric ideal MHD involve functionals of the form

$$\begin{aligned} C_F &:= \int_\Omega r^{-1} f\Phi(\Psi)\, dr\, dz, \\ C_M &:= \int_\Omega r p^{1/\gamma}\, \Phi(\Psi)\, dr\, dz, \end{aligned} \tag{2.22}$$

where $\Phi(s)$ is *any* real function (with suitable regularity properties) in the interval $\sigma_0 \leqslant s < +\infty$. The significance of these functionals derives from the fact that they can be evolved in time without explicit reference to the velocity field **V**; specifically,

$$\frac{dC_F}{dt} = 0, \qquad \frac{dC_M}{dt} = \int_\Omega rQ\Phi(\Psi)\, dr\, dz. \tag{2.23}$$

The verification of these identities is straightforward using Eqs. (2.14), (2.16), and (2.17). Of course, they represent the conservation of toroidal flux and mass within the magnetic surfaces, as can be seen by applying the formula

$$\int_\Omega a\Phi(\Psi)\, dr\, dz = \int_{\sigma_0}^{+\infty} \Phi'(\sigma)\, d\sigma \int_{\{\Psi > \sigma\}} a\, dr\, dz + \Phi(\sigma_0) \int_{\{\Psi > \sigma_0\}} a\, dr\, dz,$$

valid for any integrable function $a = a(r, z)$, to represent the functionals C_F and C_M in terms of the toroidal flux and mass within each magnetic surface, namely,

$$\int_{\{\Psi > \sigma\}} r^{-1} f\, dr\, dz, \qquad \int_{\{\Psi > \sigma\}} r p^{1/\gamma}\, dr\, dz \quad (\sigma \geqslant \sigma_0). \tag{2.24}$$

(The term "mass" is used in a loose sense, since the density $p^{1/\gamma} = e^{\zeta/\gamma\rho}\rho$ also involves the entropy.)

In addition to the above constraints of motion within the plasma, the total (plasma plus vacuum) toroidal flux,

$$F_0 := \int_\Omega r^{-1} f\, dr\, dz, \tag{2.25}$$

is conserved. In order to verify the identity

$$\frac{dF_0}{dt} = 0, \tag{2.26}$$

it is necessary to recall that the Maxwell equation

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

holds in D, while the tangential components of **E** vanish on ∂D. Then (2.26) follows immediately by applying Stokes' formula.

3. VARIATIONAL FORMULATION OF RELAXED PROBLEMS

The slow evolution equations for an axisymmetric plasma–vacuum system as given in Section 2 are degenerate in the sense that they do not include the convective derivative $D\mathbf{V}/Dt$. This set of equations is therefore underdetermined with respect to the evolution of the velocity field **V**. Nevertheless, the velocity component $V^\perp$ normal to the magnetic surfaces $\{\Psi = \sigma\}$ in the plasma, can be determined from the poloidal flux convection equation (2.16); namely, $V^\perp$ can be defined by

$$V^\perp = -|\nabla\Psi|^{-1} \frac{\partial \Psi}{\partial t}, \tag{3.1}$$

thus making (2.16) valid pointwise everywhere that the normal $\mathbf{n} = |\nabla\Psi|^{-1} \nabla\Psi$ is itself defined. The conservation laws (2.17) and (2.14) for toroidal flux and mass, on the other hand, have ambiguous meaning since each of them also involves the tangential velocity to the magnetic surfaces. A natural way to rectify this degeneracy is to relax the requirement that these equations hold at every point in the plasma to the weaker requirement that toroidal flux and mass be conserved within every magnetic surface. By relaxing these equations in such a manner we achieve two goals. First, we obtain a self-consistent formulation of the slow evolution problem (involving only $V^\perp = \mathbf{n} \cdot \mathbf{V}$). Second, we arrive at a quasi-equilibrium problem which at each instant of time has a natural variational structure.

In order to formulate the above mentioned relaxation we introduce the functionals

$$F_\sigma = \int_\Omega r^{-1} f(\Psi - \sigma)_+\, dr\, dz, \qquad M_\sigma = \int_\Omega r p^{1/\gamma} (\Psi - \sigma)_+\, dr\, dz \tag{3.2}$$

parameterized by the flux variable σ which runs through the range of Ψ in the plasma. Of course, these functionals are just the constraints of motion C_F and C_M corresponding to the particular choice $\Phi(s) = (s - \sigma)_+ := \max(s - \sigma, 0)$. Moreover, they are identical with (minus) the σ-antiderivatives of the classical quantities displayed in (2.24). In terms of these functionals we can express the relaxation of the conservation laws (2.17) and (2.14), respectively, as the equations

$$\frac{dF_\sigma}{dt} = 0, \qquad \frac{dM_\sigma}{dt} = \int_\Omega rQ(\Psi - \sigma)_+\, dr\, dz. \tag{3.3}$$

Also, because of the free-boundary, we impose Eq. (2.26) on the functional F_0 defined in (2.25). The precise formulation

of the relaxed slow evolution problem can now be stated: the plasma region is governed by the (equilibrium) force balance equations (2.19), the magnetic surface convection equation (2.16), the integral conservation laws (3.3), and the equation of state (2.7); the vacuum region is governed by the field equations (2.18); and, the fixed and free boundary conditions are given by (2.20) and (2.21).

The above relaxation from pointwise conservation laws to integral constraints of motion can be interpreted as averaging over magnetic surfaces [10, 13]. Letting

$$\langle a \rangle := \int_{\{\Psi=\sigma\}} \frac{ra}{|\nabla \Psi|}\, dl$$

denote the surface average (apart from a normalization) of a function $a = a(r, z)$, we claim that (3.3) is equivalent to

$$\frac{\partial}{\partial t}\langle r^{-2} f \rangle = 0, \qquad \frac{\partial}{\partial t}\langle p^{1/\gamma} \rangle = \langle Q \rangle,$$

where the surface averaged quantities involved depend upon σ and t. Indeed, these equations are identical with the evolution equations for F_σ and M_σ after applying $\partial^2/\partial\sigma^2$ to each of the identities in (3.3). Consequently, the relaxed slow evolution problem is identical with the surface averaged Grad–Hogan equations. We prefer, however, to replace the concept of surface averaging by the concept of integral constraints of motion in order to expose the natural variational structure of the problem.

The physical justification for averaging over magnetic surfaces can be summarized as follows. The plasma region is foliated by toroidal magnetic surfaces on which the helical field lines wind so that, at least generically, each field line is dense in its associated surface. Therefore, transport processes (see [14]) beyond the scope of the governing equations under consideration effectively enforce the postulated averaging in reality. On this basis the relaxed (averaged) equations constitute a realistic model of adiabatically slow evolution. On the same grounds we may assume that the external heating source is a surface quantity $Q = Q(\Psi, t)$, although this simplification is not strictly necessary.

We now proceed to give a variational method of solving the relaxed slow evolution problem. The prescribed data for this problem consist of the source $Q(\sigma, t)$ and the flux function $\bar{\psi}(r, z, t)$ for the poloidal field induced by the external current density $\bar{J}_\phi(r, z, t)$ according to

$$L\bar{\psi} = r\bar{J}_\phi \quad \text{in } \Omega, \qquad \bar{\psi} = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

The total flux function Ψ can be split into the sum $\Psi = \psi + \bar{\psi}$, where ψ denotes the flux function induced by current density supported in the plasma region. In what follows, ψ is used as an unknown rather than Ψ, since $\bar{\psi}$ is prescribed. The total potential energy

$$E(\psi, f, p) = \tfrac{1}{2}\int_\Omega [r^{-1}|\nabla\psi|^2 + r^{-1}f^2 + rp/(\gamma-1)]\, dr\, dz \tag{3.5}$$

serves as the objective functional; the constraint functionals are supplied by F_0, F_σ, M_σ defined above, which now can be considered as functionals of (ψ, f, p) depending explicitly on t through $\bar{\psi}(t)$. At every instant of time t we let (ψ^*, f^*, p^*) denote the solution of the constrained minimization problem

$$\begin{aligned} &E(\psi, f, p) \to \min \qquad \text{subject to} \\ &F_0(f) = F_0^*, \qquad F_\sigma(\psi, f; t) = F_\sigma^*, \\ &M_\sigma(\psi, p; t) = M_\sigma^* \end{aligned} \tag{3.6}$$

corresponding to instantaneous constraint values F_0^*, F_σ^*, M_σ^*. In doing so we assume that such a minimizer is unique (at least locally along a trajectory in the solution space), even though a general uniqueness theorem is not available. Since the validity of this assumption can be verified computationally, we can consider the formal solution map

$$(F_0^*, F_\sigma^*, M_\sigma^*) \to (\psi^*, f^*, p^*) \tag{3.7}$$

to be well defined for the purposes of discussion. With this map in hand we are able to pose the relaxed slow evolution problem as the family of equations:

$$\begin{aligned} &\frac{dF_0^*}{dt} = 0, \qquad \frac{dF_\sigma^*}{dt} = 0, \\ &\frac{dM_\sigma^*}{dt} = \int_\Omega rQ(\psi^* + \bar{\psi}, t)(\psi^* + \bar{\psi} - \sigma)_+\, dr\, dz, \end{aligned} \tag{3.8}$$

where σ runs through the range $\sigma_0 \leqslant \sigma \leqslant \max(\psi + \bar{\psi})$, which is invariant in time. In principle, the slowly evolving solution trajectory $(\psi^*(t), f^*(t), p^*(t))$ can be advanced in time according to these equations. The fact that at each instant of time t such a solution formally satisfies the MHD equilibrium equations is a consequence of the variational principle of Kruskal and Kulsrud [20].

The instantaneous variational problem (3.6) that determines $(\psi^*(t), f^*(t), p^*(t))$ is novel from the standpoint of optimization theory in the sense that it involves minimization subject to *continuously infinite* families of nonlinear and nonlocal constraints. Consequently, the calculation of the variational equations (a Lagrange multiplier rule) satisfied by a solution is not straightforward. Indeed, no rigorous derivation of the Grad–Shafranov equation (2.19) is

available in this context, because the regularity properties of the resulting profile functions for f and p (the multipliers) are not known [6, 22, 23]. In principle, an adiabatic evolution can result in very singular current density even though it is initialized by a smooth equilibrium and driven by smooth and slowly varying sources and external fields. In other words, the (instantaneous) formal solution map (3.7) does not have established continuity properties with respect to appropriate choices of function spaces for its domain and range. For this reason, we prefer to formulate a relaxed version of the above quasi-equilibrium problem in which the classical families of constraints are replaced by corresponding *finite* families by a simple interpolation procedure. In this way, we arrive at a problem that is tractable analytically and computationally.

As a technical convenience in all that follows we make the substitution

$$g := \left(\frac{2p}{\gamma - 1}\right)^{1/2} \tag{3.9}$$

and we define the functional

$$G_\sigma(\psi, g) := \int_\Omega r g^{2/\gamma} (\psi + \bar{\psi} - \sigma)_+ \, dr\, dz \tag{3.10}$$

which (up to a constant factor) is identical with the functional $M_\sigma(\psi, p)$. Then we can pose the variational problem governing the solution at every instant of time in the form

$$(P_\infty) \begin{cases} E(\psi, f, g) \to \min & \text{subject to} \\ F_0(f) = F_0^*, & F_\sigma(\psi, f) = F_\sigma^*, \\ G_\sigma(\psi, g) = G_\sigma^* & (\sigma \geqslant \sigma_0) \end{cases}$$

with the energy functional

$$E(\psi, f, g) = \tfrac{1}{2} \int_\Omega [r^{-1} |\nabla \psi|^2 + r^{-1} f^2 + r g^2] \, dr\, dz. \tag{3.11}$$

We introduce the new unknown g so that the objective functional E is expressed as a quadratic form, since this structure is needed later in the development of the iterative algorithm.

We now construct the multiconstrained variational problem that we call (P_n), whose constraints constitute a natural discretization of those in (P_∞). Let $\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n < \sigma_{n+1} = +\infty$ be a partition of the range of the flux function $\Psi = \psi + \bar{\psi}$ in the plasma with $\sigma_0 = \min \Psi$ and $\sigma_n = \max \Psi$; and let $\Delta\sigma_i := \sigma_i - \sigma_{i-1}$. We define "basis functions" relative to this partition by

$$\Phi_i(s) := \frac{1}{\Delta\sigma_i} \int_{\sigma_{i-1}}^{\sigma_i} (s - \sigma)_+ \, d\sigma - \frac{1}{\Delta\sigma_{i+1}} \int_{\sigma_i}^{\sigma_{i+1}} (s - \sigma)_+ \, d\sigma, \tag{3.12}$$

or, explicitly,

$$\Phi_i(s) = \begin{cases} 0, & s \leqslant \sigma_{i-1} \\ (s - \sigma_{i-1})^2 / 2\Delta\sigma_i, & \sigma_{i-1} \leqslant s \leqslant \sigma_i \\ s - (s - \sigma_i)^2 / 2\Delta\sigma_{i+1} - (\sigma_{i-1} + \sigma_i)/2, & \sigma_i \leqslant s \leqslant \sigma_{i+1} \\ (\sigma_{i+1} - \sigma_{i-1})/2, & s \geqslant \sigma_{i+1}. \end{cases}$$

These functions are precisely the s-antiderivatives of the usual finite element functions $\Phi_i'(s)$; namely, for $1 \leqslant i \leqslant n-1$, Φ_i' is piecewise-linear on $\sigma_{i-1} \leqslant s \leqslant \sigma_{i+1}$, zero elsewhere, and $\Phi_i'(\sigma_i) = 1$, while Φ_n' is linear on $\sigma_{n-1} \leqslant s \leqslant \sigma_n$, constant on $s \geqslant \sigma_n$, and $\Phi_n'(\sigma_n) = 1$. In terms of these basis functions we define the constraint functionals

$$F_i(\psi, f) = \int_\Omega r^{-1} f \Phi_i(\psi + \bar{\psi}) \, dr\, dz \tag{3.13}$$

$$G_i(\psi, g) = \int_\Omega r g^{2/\gamma} \Phi_i(\psi + \bar{\psi}) \, dr\, dz. \tag{3.14}$$

Now we can pose the relaxed variational problem

$$(P_n) \begin{cases} E(\psi, f, g) \to \min & \text{subject to} \\ F_0(f) = F_0^*, & F_i(\psi, f) = F_i^*, \\ G_i(\psi, g) = G_i^* & (i = 1, \ldots, n), \end{cases}$$

whose constraint values are derived from those in (P_∞) according to

$$F_i^* = \frac{1}{\Delta\sigma_i} \int_{\sigma_{i-1}}^{\sigma_i} F_\sigma^* \, d\sigma - \frac{1}{\Delta\sigma_{i+1}} \int_{\sigma_i}^{\sigma_{i+1}} F_\sigma^* \, d\sigma$$

and similarly for G_i^*.

The variational equations associated with (P_n) are readily derived. Let F' denote the functional (or Fréchet) derivative of any differentiable functional F in the triple (ψ, f, g). The standard Lagrange multiplier rule [15, 25] states that a solution (ψ, f, g) of (P_n) satisfies

$$E'(\psi, f, g) = \lambda_0 F_0'(f) + \sum_{i=1}^{n} \lambda_i F_i'(\psi, f) + \mu_i G_i'(\psi, g) \tag{3.15}$$

for some multipliers $\lambda_0, \lambda_i, \mu_i \in \mathbb{R}$. (Technically, these multipliers exist and are uniquely determined by the minimizer, provided the constraint derivatives are linearly independent. This condition can be verified in the present

context.) An explicit calculation of the expressions entering into (3.15) yields the triple of equations

$$L\psi = f\lambda \cdot \Phi'(\psi + \bar{\psi}) + r^2 g^{2/\gamma} \mu \cdot \Phi'(\psi + \bar{\psi}) \tag{3.16}$$

$$f = \lambda_0 + \lambda \cdot \Phi(\psi + \bar{\psi}) \tag{3.17}$$

$$g = (2/\gamma)\, g^{(2-\gamma)/\gamma} \mu \cdot \Phi(\psi + \bar{\psi}), \tag{3.18}$$

where the abbreviated notation $\lambda \cdot \Phi = \sum_{i=1}^{n} \lambda_i \Phi_i$ is used. These equations are just a version of the Grad–Shafranov equation (2.19). Indeed, it is immediate that (3.17) defines the profile $f = f(\Psi)$; and it follows after some manipulation that (3.18) defines the profile $p = p(\Psi) = (\gamma - 1)\, g(\Psi)^2/2$. Thus, the (unknown) profile functions are determined by the multipliers λ_0, λ_i, μ_i according to

$$\begin{aligned} f(\Psi) &= \lambda_0 + \lambda \cdot \Phi(\Psi), \\ p(\Psi) &= \frac{(\gamma - 1)}{2} \left[\frac{2}{\gamma} \mu \cdot \Phi(\Psi) \right]^{\gamma/(\gamma - 1)}. \end{aligned} \tag{3.19}$$

In view of these relations, Eqs. (3.16)–(3.18) are seen to be equivalent to the classical equilibrium equations (2.6) and (2.9) in the plasma and vacuum regions. Moreover, the free-boundary conditions on $\mathcal{S} = \{\Psi = \sigma_0\}$ are also valid, because the continuity (and differentiability) of $\nabla\psi$, f, and g are implied by these equations, by virtue of standard elliptic regularity theory. In summary, *any minimizer* (ψ, f, g) *for* (P_n) *produces an exact solution of the complete equilibrium problem for the plasma–vacuum system.*

The variational problem (P_n) can be viewed as a particular formulation of the general variational principle for ideal magnetohydrodynamic equilibria due to Woltjer [27, 28]. In his principle the constraints, which are derived as invariants of the associated evolution equations, take exactly the same form as our functionals C_F and C_M. A derivation of the associated variational equations is included in his work. Our presentation differs from his, however, in the choice of specific basic functions Φ_i, which are tailored to numerical analysis, and in the complete treatment of the plasma–vacuum free-boundary problem for a toroidal confinement device. The development we give in this section also connects the Kruskal–Kulsrud principle with the Woltjer principle by means of the idea that the constraints for (P_∞), which are parametrized by the magnetic surface variable σ, are naturally discretized into the constraints for (P_n), which are parametrized by the index i. In a sense this discretization of constraints is achieved by averaging the σ-parametrized constraints over subintervals of the partition $\{\sigma_i\}$. Therefore, as the class of admissible triples (ψ, f, g) is enlarged in going from (P_∞) to (P_n), the solutions of (P_n) are themselves *exact* solutions of the governing equilibrium equations. Their associated profile functions $f(\Psi)$ and $p(\Psi)$, however, are special in that they are constructed from linear combinations of the basis functions $\Phi_i(\Psi)$, $i = 1, \ldots, n$. As n tends to infinity, these interpolating profile functions tend to the general profile functions appropriate to (P_∞), and so a natural approximation of the surface-averaged quasi-equilibrium problem is obtained.

The relaxed slow evolution problem corresponding to (P_n) is simply a $(2n+1)$-dimensional system of ordinary differential equations. Introducing the vector of constraint values

$$X^* := (F_0^*, F_i^*, M_i^*) \in \mathbb{R}^{2n+1},$$

this system is expressible as

$$\frac{dX^*}{dt} = A(X^*, t), \tag{3.20}$$

where A has the components $(i = 1, \ldots, n)$

$$\begin{aligned} A_0 &= 0, \qquad A_i = 0 \\ A_{n+i} &= \int_\Omega rQ(\psi^* + \bar{\psi}, t)\, \Phi_i(\psi^* + \bar{\psi})\, dr\, dz. \end{aligned} \tag{3.21}$$

Here, (ψ^*, f^*, g^*) denotes the minimizer for (P_n) corresponding to the constraint values X^*, which is assumed to be uniquely determined by X^*, so the evaluation of A requires the solution of (P_n). Thus, our relaxed (volume-averaged) version of the nondissipative Grad–Hogan equations is rigorously formulated, and the dynamical system (3.20) furnishes a complete and consistent model of the slow evolution of plasma–vacuum system due to heating sources and/or varying external fields.

It remains to discuss the initial conditions for the system (3.20). In most realistic situations, the external sources or currents that drive the slow evolution are applied to a known equilibrium configuration of interest. For instance, this is the case when either plasma heating on adiabatic compression is initiated. Moreover, slow evolution can be expected only if the initial configuration is almost in equilibrium. We shall assume, therefore, that the initial constraint values $X^*(0)$ are derived from a given equilibrium solution.

4. ITERATIVE ALGORITHM

We now construct an iterative procedure that solves the variational problem (P_n) formulated in the preceding section. In this section we develop the abstract form of the algorithm and derive its general convergence properties. In the succeeding section we state the algorithm in its concrete form and discuss its implementation for toroidal equilibrium computations.

A preliminary transformation of (P_n) into an equivalent variational problem is required before the algorithm given for $(\tilde{P}_n)$ is now rewritten with the functionals $F_0, ..., F_{2n}$ replacing $\tilde{F}_0, F_1, ..., F_n, G_1, ..., G_n$ and with corresponding

modified variational problem

$$(\tilde{P}_n)\begin{cases} E(\psi, f, g) \to \min & \text{subject to} \\ \tilde{F}_0(\psi, f) = \tilde{F}_0^*, & F_i(\psi, f) = F_i^*, \\ G_i(\psi, g) = G_i^* & (i = 1, ..., n), \end{cases}$$

where the total toroidal flux constraint is replaced by a constraint imposed on the functional

$$\tilde{F}_0(\psi, f) = F_0(f) - \sum_{i=1}^{n} \omega_i F_i(\psi, f) \tag{4.1}$$

defined by some constants $\omega_i \geqslant 0$. This minor change in (P_n) is necessary because the structure of the algorithm rests on the property that the (Lagrange) multipliers corresponding to a minimizer are *strictly positive*. However, under typical circumstances, the multipliers for (P_n) are expected to have the signs $\lambda_0 > 0$, $\lambda_i < 0$, $\mu_i > 0$. Indeed, apart from the sign of λ_0, which is set by the direction of the toroidal magnetic field in the vacuum, the signs of these multipliers can be inferred from the monotonicity of the profile functions in (3.19)—namely, from the inequalities $f'(\Psi) < 0$, $p'(\Psi) > 0$, which are anticipated on physical grounds. This is so by virtue of the choice of the basis functions Φ_i as antiderivatives of the finite element functions, combined with the formulas

$$\begin{aligned} f'(\Psi) &= \lambda \cdot \Phi'(\Psi), \\ p'(\Psi) &= \left[\frac{2}{\gamma}\mu \cdot \Phi(\Psi)\right]^{1/(\gamma-1)} \mu \cdot \Phi'(\Psi). \end{aligned} \tag{4.2}$$

Now, if $\tilde{\lambda}_0, \tilde{\lambda}_i, \tilde{\mu}_i$ denote the multipliers for $(\tilde{P}_n)$, then they are related to those for (P_n) by

$$\lambda_0 = \tilde{\lambda}_0, \ \lambda_i = \tilde{\lambda}_i - \lambda_0 \omega_i, \ \mu_i = \tilde{\mu}_i \qquad (i = 1, ..., n), \tag{4.3}$$

as is immediate from the comparison of the variational equations for (P_n) with their analogues for $(\tilde{P}_n)$. Consequently, the constants $\omega_1, ..., \omega_n$ can always be fixed large enough (depending upon the solution) to ensure that all of the multipliers for $(\tilde{P}_n)$ are strictly positive. Throughout the remainder of the section we assume that this is done.

For the sake of simplicity of exposition we let $u = (\psi, f, g)$ denote the unknown triple, and we write the variational problem $(\tilde{P}_n)$ in the abstract form

$$E(u) \to \min \qquad \text{subject to} \quad F_i(u) = F_i^* \quad (i = 0, ..., 2n). \tag{4.4}$$

In an obvious change of notation, the constraint family belongs to the space $H = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, for which a norm $\|\cdot\|_H$ is given by

$$\|u\|_H^2 = \int_\Omega [|\nabla\psi|^2 + f^2 + g^2] \, dr \, dz.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard L^2-pairing in the sense of distribution theory—namely,

$$\langle u_1, u_2 \rangle := \int_\Omega [\psi_1\psi_2 + f_1 f_2 + g_1 g_2] \, dr \, dz,$$

where $f_1, f_2, g_1, g_2 \in L^2(\Omega)$, and $\psi_1 \in H_0^1(\Omega)$, $\psi_2 \in H^{-1}(\Omega)$ or vice versa. In terms of this pairing the objective functional E, which is differentiable at any $u \in H$, has the derivative

$$E'(u) = (r^{-1}L\psi, r^{-1}f, rg) \in H^{-1}(\Omega) \times L^2(\Omega) \times L^2(\Omega),$$

meaning that $E(u + \delta u) = E(u) + \langle E'(u), \delta u \rangle + o(\|\delta u\|_H)$ as δu tends to zero in H. Similar remarks pertain to the constraint functionals F_i. The variational equations satisfied by a minimizer $u \in H$, interpreted in this sense, are simply

$$E'(u) = \sum_{j=0}^{2n} \lambda_j F_j'(u) \tag{4.5}$$

with positive multipliers $\lambda_0, ..., \lambda_{2n}$, which are identified with the multipliers in $(\tilde{P}_n)$.

Convexity properties of the objective and constraint functionals are fundamental to the construction of the iterative algorithm. The objective functional E is both *strictly convex* and *quadratic*; it admits the useful expansion

$$E(v) = E(u) + \langle E'(u), v - u \rangle + E(v - u), \tag{4.6}$$

in which the positive second-order term is explicit. On the other hand, the constraint functionals F_i are not convex. Therefore, the algorithm invokes a certain "convexification" of them, and requires that constants $\alpha_i \geqslant 0$ be fixed such that

$$F_i + \alpha_i E \qquad \text{is convex} \qquad (i = 0, ..., 2n). \tag{4.7}$$

Clearly, α_i can be chosen large enough to guarantee this property. With $\alpha_0, ..., \alpha_{2n}$ fixed appropriately, the algorithm can now be stated.

Let $u^0 \in H$ satisfying $F_i(u^0) \geqslant F_i^*$ $(i = 0, ..., 2n)$ be an (otherwise arbitrary) initialization. The iterative sequence u^k, $k = 0, 1, ...,$ is defined inductively by solving the

following *quadratic programming subproblem* at the iterative step: $u = u^{k+1}$ solves

$$E(u) \to \min \qquad \text{subject to}$$
$$F_i(u^k) + \langle F_i'(u^k) + \alpha_i E'(u^k), u - u^k \rangle \geqslant F_i^* \qquad (4.8)$$
$$(i = 0, ..., 2n).$$

This subproblem is a convex optimization problem having linear inequality constraints. Therefore, it has a unique solution which is characterized by the corresponding Kuhn–Tucker conditions (the analogue of the Lagrange multiplier rule) [15, 25]; namely, $u^{k+1} \in H$ and its associated Kuhn–Tucker (multiplier) vector $\xi^{k+1} \in \mathbb{R}^{2n+1}$ are uniquely determined by

$$E'(u^{k+1}) = \sum_{j=0}^{2n} \xi_j^{k+1} [F_j'(u^k) + \alpha_j E'(u^k)] \qquad (4.9)$$

$$\xi_i^{k+1} \geqslant 0 \qquad (4.10)$$

$$F_i(u^k) + \langle F'(u^k) + \alpha_i E'(u^k), u^{k+1} - u^k \rangle - F_i^* \geqslant 0 \qquad (4.11)$$

$$\xi_i^{k+1} [F_i(u^k) + \langle F_i'(u^k) + \alpha_i E'(u^k), u^{k+1} - u^k \rangle - F_i^*] = 0. \qquad (4.12)$$

An explicit description of the algorithm defined by (4.8) can be given in terms of the so-called dual subproblem [15, 25]. Let a matrix (a_{ij}^k) and a vector (c_i^k) be defined by the iterate u^k according to

$$a_{ij}^k = \langle F_i'(u^k) + \alpha_i E'(u^k), M^{-1}[F_j'(u^k) + \alpha_j E'(u^k)] \rangle \qquad (4.13)$$

$$c_i^k = F_i^* - F_i(u^k) + \langle F_i'(u^k) + \alpha_i E'(u^k), u^k \rangle, \qquad (4.14)$$

where $M = E''$ denotes the (unbounded) symmetric linear operator that corresponds to the positive definite quadratic form

$$E(u) = \tfrac{1}{2} \langle Mu, u \rangle \qquad (u \in H). \qquad (4.15)$$

The subproblem in ξ that is dual to (4.8) is then expressible as

$$\tfrac{1}{2} \sum_{i,j=0}^{2n} a_{ij}^k \xi_i \xi_j - \sum_{i=0}^{2n} c_i^k \xi_i \to \min \qquad \text{over} \quad \xi_i \geqslant 0. \qquad (4.16)$$

Now the algorithm can be described in a form that leads directly to a concrete numerical implementation. The iterative step is defined by

$$u^{k+1} = \sum_{j=0}^{2n} \xi_j^{k+1} M^{-1} [F_j'(u^k) + \alpha_j E'(u^k)], \qquad (4.17)$$

where ξ^{k+1} is the (unique) solution of (4.16). The equivalence of (4.16), (4.17) with (4.8) is standard. The variational inequalities satisfied by ξ^{k+1} are simply

$$\sum_{j=0}^{2n} a_{ij}^k \xi_j^{k+1} - c_i^k \begin{cases} \geqslant 0 & \text{if } \xi_i^{k+1} = 0 \\ = 0 & \text{if } \xi_i^{k+1} > 0. \end{cases} \qquad (4.18)$$

Noting that (4.17) coincides with (4.9) by virtue of the identity $E'(u) = Mu$, and substituting this expression for u^{k+1} into the complementarity conditions (4.10)–(4.12), it is evident that (4.18) is just a restatement of those conditions.

We now turn to a discussion of the convergence properties of the algorithm. These special properties dictate the structure of the algorithm, and hence justify the particular construction described above. It is essential to explain them in conjunction with the definition of the iterative algorithm.

In short, the algorithm defined in (4.8) possesses the monotonicity properties:

$$E(u^{k+1}) \leqslant E(u^k), \qquad (4.19)$$

$$F_i(u^k) \geqslant F_i^* \qquad (i = 0, ..., 2n) \qquad (4.20)$$

for every k and *any* admissible initialization u^0. An inductive argument on k proves this claim. Suppose that $F_i(u^k) \geqslant F_i^*$. Then u^k is admissible in the subproblem (4.8) which defines u^{k+1}, and hence (4.19) follows. Since $F_i + \alpha_i E$ is convex by hypothesis, there holds

$$\begin{aligned} F_i(u^{k+1}) + \alpha_i E(u^{k+1}) &\geqslant F_i(u^k) + \alpha_i E(u^k) \\ &\quad + \langle F_i'(u^k) + \alpha_i E'(u^k), u^{k+1} - u^k \rangle \\ &\geqslant F_i^* + \alpha_i E(u^k), \end{aligned} \qquad (4.21)$$

invoking the linear inequality constraints in (4.8). Now (4.21) and (4.19) combine to give $F_i(u^{k+1}) \geqslant F_i^*$, completing the induction.

A stronger version of the monotonicity property (4.19) is true; namely,

$$E(u^k - u^{k+1}) \leqslant E(u^k) - E(u^{k+1}), \qquad (4.22)$$

for every k. This inequality can be derived from the Kuhn–Tucker conditions (4.9)–(4.12), along with the expansion (4.6). The required calculation is

$$\begin{aligned} E(u^k) - E(u^{k+1}) &= \langle E'(u^{k+1}), u^k - u^{k+1} \rangle + E(u^k - u^{k+1}) \\ &= \sum \xi_j^{k+1} \langle F_j'(u^k) + \alpha_j E'(u^k), u^k - u^{k+1} \rangle \\ &\quad + E(u^k - u^{k+1}) \\ &\geqslant \sum \xi_j^{k+1} [F_j(u^k) - F_j^*] + E(u^k - u^{k+1}) \\ &\geqslant E(u^k - u^{k+1}). \end{aligned} \qquad (4.23)$$

From (4.22) both the monotonic convergence of the objective functional values,

$$E(u^k) \downarrow E^* \quad \text{(say)} \quad \text{as} \quad k \to \infty,$$

and the convergence property of the iterative sequence,

$$\|u^k - u^{k+1}\|_H \to 0 \quad \text{as} \quad k \to \infty, \tag{4.24}$$

follow. Without a rate for this convergence it cannot be asserted that the entire sequence $\{u^k\}$ converges to a single limit point $u^* \in H$. Nevertheless, the bound $E(u^k) \leqslant E(u^0)$ ensures at least that every subsequence has a furher subsequence converging weakly in H. Now, if u^* denotes such a weak limit point, then a straightforward argument (which is omitted here) furnishes the corresponding weak convergence of the Kuhn–Tucker conditions; namely, the associated subsequence of multiplier vectors ξ_j^k tends to a limit ξ^*, and the pair u^*, ξ^* satisfies

$$\langle E'(u^*), v\rangle = \sum_{j=0}^{2n} \xi_j^* \langle F_j'(u^*) + \alpha_j E'(u^*), v\rangle \quad \text{for all} \quad v \in H, \tag{4.25}$$

$$\xi_i^* \geqslant 0 \tag{4.26}$$

$$F_i(u^*) - F_i^* \geqslant 0 \tag{4.27}$$

$$\xi_i^* [F_i(u^*) - F_i^*] = 0. \tag{4.28}$$

But (4.25) is just the weak form of the governing variational equations (4.5) when the multipliers λ_i^* associated with u^* are defined by

$$\lambda_i^* = \xi_i^*/(1 - \alpha \cdot \xi^*). \tag{4.29}$$

Thus, u^* is a critical point (a minimum point) for the variational problem (4.4) whenever (a) $\xi_i^* > 0$ for all i, and (b) $\alpha \cdot \xi^* < 1$. As remarked above, the choice of ω guarantees that $\lambda_i^* > 0$ for all i, which in view of condition (b) is the same as condition (a). Hence the complementarity conditions (4.28) enforce the *equality* constraints in (4.4) at the solution u^*. It suffices then to verify condition (b). This can be inferred from the identity

$$(1 - \alpha \cdot \xi^*)\langle E'(u^*), u^*\rangle = \sum \xi_j^* \langle F_j'(u^*), u^*\rangle,$$

which results from taking $v = u^*$ in (4.25). Under mild conditions, $\langle E'(u^*), u^*\rangle > 0$ and $\langle F_j^*(u^*), u^*\rangle > 0$, and so condition (b) follows.

The convergence properties demonstrated above refer to subsequences of the iterative sequence $\{u^k\}$. An alternative statement of these makes use of the distance in the space H between a point $u \in H$ and the set $S^* \subseteq H$ of solutions (critical points) of the variational problem (4.4). It can be shown that as $k \to \infty$

$$\text{dist}_H(u^k, S^*) := \inf\{\|u^k - u\|_H : u \in S^*\} \to 0.$$

This generalized form of the convergence statement is required because of the possibility that solutions of (4.4) are not unique. As the proof is mainly of technical interest, we omit it here. The reader is referred to our earlier work [6] for detailed convergence proofs in the case of a prototype problem for (P_n).

In implemented computations such as those documented in Section 6 the iterative sequence is observed to converge to a *single* limit point with a *linear* rate of convergence. This convergence behavior is expected whenever the algorithm is applied to compute a (local) energy minimizing solution subject to the given flux and mass constraints. Indeed, the convergence rate can be related to the stability of the equilibrium to *axisymmetric* perturbations.

5. NUMERICAL IMPLEMENTATION

The concrete form of the iterative algorithm defined in Section 4 is presented next. For this purpose we first record in component form some of the expressions used in the abstract statement of the algorithm. For the sake of simplicity we choose $\gamma = 2$ here and in the computations discussed in Section 6. The objective functional E given by (4.15) has the derivative

$$Mu = E'(u) = (r^{-1}L\psi, r^{-1}f, rg). \tag{5.1}$$

The constraint functionals $F_0, ..., F_{2n}$, which are identified with $\tilde{F}_0, F_1, ..., F_n, G_1, ..., G_n$, have the derivatives

$$\begin{aligned} F_0'(u) &= (r^{-1}f\tilde{\Phi}_0'(\psi + \bar{\psi}), r^{-1}\tilde{\Phi}_0(\psi + \bar{\psi}), 0) \\ F_i'(u) &= (r^{-1}f\Phi_i'(\psi + \bar{\psi}), r^{-1}\Phi_i(\psi + \bar{\psi}), 0) \\ F_{n+i}'(u) &= (rg\Phi_i'(\psi + \bar{\psi}), 0, r\Phi_i(\psi + \bar{\psi})) \end{aligned} \tag{5.2}$$

for $i = 1, ..., n$. The basis function $\tilde{\Phi}_0(s) := 1 - \omega \cdot \Phi(s)$ is included so that the functional $\tilde{F}_0$ may be expressed in the form

$$\tilde{F}_0(u) = \int_\Omega r^{-1}f\tilde{\Phi}_0(\psi + \bar{\psi})\, dr\, dz.$$

Some functionals derived from these expressions are introduced for convenience,

$$P_{ij}(u) := \langle F_i'(u), M^{-1}F_j'(u)\rangle \tag{5.3}$$

$$Q_i(u) := \langle F_i'(u), u\rangle, \tag{5.4}$$

for $i = 0, ..., 2n$. According to (5.1), the terms involving M^{-1} are calculated to be

$$\begin{aligned} M^{-1}F_0'(u) &= (w_0, \tilde{\Phi}_0(\psi + \bar{\psi}), 0) \\ M^{-1}F_i'(u) &= (w_i, \Phi_i(\psi + \bar{\psi}), 0) \\ M^{-1}F_{n+i}'(u) &= (w_{n+i}, 0, \Phi_i(\psi + \bar{\psi})), \end{aligned} \tag{5.5}$$

where $w_0, ..., w_{2n}$ are determined by solving the linear elliptic boundary value problems:

$$\begin{aligned} Lw_0 &= f\tilde{\Phi}_0'(\psi + \bar{\psi}) && \text{in } \Omega, \\ w_0 &= 0 && \text{on } \partial\Omega \\ Lw_i &= f\Phi_i'(\psi + \bar{\psi}) && \text{in } \Omega, \\ w_i &= 0 && \text{on } \partial\Omega \\ Lw_{n+i} &= r^2 g\Phi_i'(\psi + \bar{\psi}) && \text{in } \Omega, \\ w_{n+i} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.6}$$

The iterative step that generates $u^{k+1} = (\psi^{k+1}, f^{k+1}, g^{k+1})$ from $u^k = (\psi^k, f^k, g^k)$ can be described as the three stage process:

Stage 1. Find the solutions $w_0^k, ..., w_{2n}^k$ of (5.6) corresponding to (ψ^k, f^k, g^k).

Stage 2. Evaluate the integrals $E(u^k)$, $F_i(u^k)$, $P_{ij}(u^k)$, $Q_i(u^k)$ $(i, j = 0, ..., 2n)$; then assemble the coefficients

$$\begin{aligned} a_{ij}^k &= P_{ij}(u^k) + \alpha_i Q_j(u^k) \\ &\quad + \alpha_j Q_i(u^k) + 2\alpha_i\alpha_j E(u^k) \end{aligned} \tag{5.7}$$

$$c_i^k = F_i^* - F_i(u^k) + Q_i(u^k) + 2\alpha_i E(u^k), \tag{5.8}$$

and solve the dual quadratic programming subproblem (4.16) for ξ^{k+1}.

Stage 3. Set

$$\psi^{k+1} = \sum_{j=0}^{2n} \xi_j^{k+1} w_j + (\alpha \cdot \xi^{k+1})\,\psi^k \tag{5.9}$$

$$\begin{aligned} f^{k+1} &= \xi_0^{k+1}\tilde{\Phi}_0(\psi^k + \bar{\psi}) \\ &\quad + \sum_{j=1}^{n} \xi_j^{k+1}\Phi_j(\psi^k + \bar{\psi}) + (\alpha \cdot \xi^{k+1}) f^k \end{aligned} \tag{5.10}$$

$$g^{k+1} = \sum_{j=1}^{n} \xi_{n+j}^{k+1}\Phi_j(\psi^k + \bar{\psi}) + (\alpha \cdot \xi^{k+1})\, g^k. \tag{5.11}$$

Stage 1 involves the solution of $2n + 1$ Dirichlet problems in Ω. These linear problems can be treated by any standard numerical method, the structure of the algorithm being independent of the particular discretization used.

Stage 2 constructs and then solves the dual subproblem for ξ^{k+1}. The numerical integrations required to compute the coefficients given in (5.7) and (5.8) can be accomplished by any numerical quadrature appropriate to the discretization used. The determination of ξ^{k+1}, a convex quadratic minimization problem with simple inequality constraints, can be achieved by a variety of known optimization methods. In the event that none of the constraints holds as an equality, the multiplier vector ξ^{k+1} is just the solution of the linear system

$$\sum_{j=0}^{2n} a_{ij}^k \xi_j^{k+1} = c_i^k \qquad (i = 0, ..., 2n).$$

This event is common in implemented computations, since $(\tilde{P}_n)$ is formulated so that $\xi_i^* > 0$. Accordingly, an effective strategy is as follows: if the solution ξ^{k+1} of the linear system satisfies $\xi_i^{k+1} > 0$ for all i, then it is accepted as the multiplier vector; otherwise, the multiplier vector is found by solving the quadratic programming problem.

Stage 3 defines u^{k+1} in terms of u^k and ξ^{k+1} according to the explicit formula (4.17) for the solution of the primal subproblem (4.8).

The algorithm exhibited above is completely specified once the "shifting" parameters $\omega_1, ..., \omega_n$ and the "convexifying" parameters $\alpha_0, ..., \alpha_{2n}$ are fixed. The parameters ω_i depend upon the solution through $f(\Psi)$, and consequently they must be chosen adaptively to ensure the positivity of the multipliers associated with the solution of $(\tilde{P}_n)$. The choice of the parameters α_i may also be made adaptively on the basis of a posteriori information obtained in practice, but certain a priori estimates for these parameters are available in theory. In particular, lower bounds for the parameters α_i which are sufficient conditions for the convexity properties (4.7) can be inferred from the positivity of the second-order derivatives of the functionals $F_i + \alpha_i E$. For example, a calculation shows that the following a priori lower bound for α_i with $1 \leqslant i \leqslant n$ suffices:

$$\Lambda_\Omega \alpha_i \geqslant 1 + (\Delta\sigma_{i+1})^{-1} \max_{\Omega} |f| \qquad (1 \leqslant i \leqslant n),$$

where the constant Λ_Ω is the smallest eigenvalue for L in Ω. Similar lower bounds for α_i with either $i = 0$ or $n + 1 \leqslant i \leqslant 2n$ can be found. Such estimates are useful as guides to the appropriate choice of the parameters α_i. Nevertheless, they are too crude to be employed effectively in the implemented algorithm, in which the resulting monotonicity and convergence properties are observed to persist for values of the parameters α_i that are consierably smaller than these estimated values. Moreover, the performance of the algorithm improves as the parameters α_i are decreased, both with respect to the conditioning of the dual subproblem and the rate of convergence of the iterative sequence. In this regard, our computations suggest that the optimal choice of

α_i is roughly $\frac{1}{10}$ of its estimated value and that nearby choices (within a factor of 3) result in nearly the same performance. On the other hand, our tests indicate that the algorithm performs poorly when the parameters α_i are chosen too small or too large.

Finally, we briefly describe a solver for the standard equilibrium problem with prescribed toroidal current profile and total toroidal current, which we use to initialize a slow (adiabatic) evolution computation. This especially simple method of solution, which is also based on a variational formulation, is fully discussed in another paper [5], where the analogous hydrodynamic problems are treated.

The equilibrium problem for the plasma–vacuum system is to be solved with a given total toroidal current $I>0$ and given profile functions in the Grad–Shafranov equation (2.19) having the form

$$f(\Psi)=f_0((\Psi-\sigma_0)_+), \qquad p(\Psi)=p_0((\Psi-\sigma_0)_+);$$

the prescribed functions $f_0(s)$ and $p_0(s)$ are assumed to be smooth for $s\geqslant 0$ and to satisfy $f_0(0)>0$, $f_0'(0)=0$ and $p_0(0)=0$, $p_0'(0)=0$. The problem is then to find ψ and σ_0 such that

$$\begin{aligned} L\psi &= rj(r, \psi+\bar{\psi}-\sigma_0) \qquad \text{in } \Omega, \\ \psi &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{5.12}$$

$$\int_\Omega j(r, \psi+\bar{\psi}-\sigma_0)\, dr\, dz = I, \tag{5.13}$$

where the toroidal current density in the plasma is written as

$$j(r, s)=r^{-1}f_0(s_+)f_0'(s_+)+rp_0'(s_+).$$

In this problem (in contrast to (P_n)), the flux constant σ_0 is an unknown and determines the location of the plasma–vacuum interface $\{\psi+\bar{\psi}=\sigma_0\}$ in response to the total current constraint.

The iterative algorithm appropriate to this problem can be stated as follows: given ψ^k, define ψ^{k+1} and σ_0^{k+1} by

$$\begin{aligned} L\psi^{k+1} &= rj(r, \psi^k+\bar{\psi}-\sigma_0^{k+1}) \qquad \text{in } \Omega, \\ \psi^{k+1} &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{5.14}$$

$$\int_\Omega j(r, \psi^k+\bar{\psi}-\sigma_0^{k+1})\, dr\, dz = I. \tag{5.15}$$

This iterative step is easily implemented by first finding σ_0^{k+1} to satisfy (5.15) and then solving the linear elliptic boundary value problem (5.14) for ψ^{k+1}.

In the computed examples considered below, the Grad–Shafranov profiles are taken to have the form

$$f_0^2/2=a_0+a_1s_+^{\kappa+1}/(\kappa+1),$$
$$p_0=a_2s_+^{\kappa+1}/(\kappa+1), \qquad j=(a_1/r+a_2r)\,s_+^\kappa.$$

The parameters $a_0>0$, $a_1<0$, $a_2>0$, and $\kappa>0$ are chosen to obtain solutions having acceptable physical properties.

6. COMPUTED EXAMPLES

In this section we present the results of some computations made with the general algorithm described above. For the sake of brevity, we limit our discussion to two representative cases, each of which is chosen to exhibit an aspect of the performance of the algorithm. In the first case, we generate a family of flux-conserving equilibria with increasing plasma pressure by imposing a heating source. In the second case, we generate a family of adiabatically compressed equilibria in an external poloidal field by varying the toroidal coil currents.

In each case, the cross-sectional domain Ω is taken to be a rectangle, even though our general method is not restricted to this simple geometry. The operator L in Ω is discretized by a standard finite difference method, and the linear elliptic boundary-value problems constituting Stage 1 of the algorithm are treated with a corresponding fast Poisson solver. All of the integrals required in Stage 2 of the algorithm are computed using appropriate quadrature formulas consistent with the discretization. The dual quadratic programming subproblem for ξ^{k+1} is solved by a standard routine based on an active set method [9]. This method, which employs the direct linear solution of reduced systems, is precise and efficient, and it exploits some special feature of the algorithm. The explicit formulas for $(\psi^{k+1}, f^{k+1}, g^{k+1})$ which form Stage 3 of the algorithm are evaluated at each grid node. The iterations are terminated when the criteria

$$\max\left\{\frac{\|\psi^{k+1}-\psi^k\|_2}{\|\psi^k\|_2}, \frac{\|f^{k+1}-f^k\|_2}{\|f^k\|_2}, \frac{\|g^{k+1}-g^k\|_2}{\|g^k\|_2}\right\}<0.003$$

$$\max\left\{\frac{F_0^{k+1}-F_0^*}{F_0^*}, \frac{F_i^{k+1}-F_i^*}{F_i^*}, \frac{G_i^{k+1}-G_i^*}{G_i^*}\right\}<0.003,$$

are achieved, where F_0^{k+1}, F_i^{k+1}, G_i^{k+1} denote the constraints evaluated at $(\psi^{k+1}, f^{k+1}, g^{k+1})$. Among the examples given below (which are representative) between 10 and 50 iterations are required to satisfy these criteria.

The first family of computed equilibria illustrates the

evolution from low-β to high-β, the volume-averaged β being defined by

$$\beta := \int p\, dV \Big/ \int \tfrac{1}{2}\mathbf{B}^2\, dV.$$

The computational domain $\Omega = \{0.5 < r < 1.5,\ |z| < 0.5\}$ is discretized with a grid having 101×101 nodes. There are no external coils ($\bar{\psi} = 0$), so the magnetic geometry is determined simply by the boundary condition $\psi = 0$ on the conducting shell $\partial\Omega$. The quasi-equilibrium problem governed by (P_n) is considered with $n = 5$. The family is initialized by a solution of (5.12), (5.13) corresponding to

$$a_0 = 50, \quad a_1 = -5, \quad a_2 = 50, \quad \kappa = 1, \quad I = 3.5.$$

The constraint values furnished by this solution are

$$\begin{aligned} &F_0 = 10.982, \quad \sigma_0 = 0.287, \\ &F_1 = 0.195, \quad F_2 = 0.130, \quad F_3 = 0.0793, \\ &F_4 = 0.0368, \quad F_5 = 0.00585, \\ &G_1 = 0.0513, \quad G_2 = 0.0428, \quad G_3 = 0.0309, \\ &G_4 = 0.0164, \quad G_5 = 0.00284. \end{aligned} \tag{6.1}$$

These constraints correspond to equal flux increments $\Delta\sigma_i = (\max \psi - \sigma_0)/5$. With these values defining $X^*(0)$, the vector of constraint values $X^*(t)$ is evolved in slow time t according to (3.20) with a heating source given by

$$Q = 40(\psi - \sigma_0)^2.$$

The solution of (3.20) is advanced with a time step $\Delta t = 0.05$ using the standard fourth-order Runge–Kutta scheme; at each intermediate step the function $A(X)$ is evaluated by solving (P_5) with constraints X.

Table I records the computed results for this adiabatic evolution on the time interval $0 \leqslant t \leqslant 1$. It lists the mass constraints G_i^* which increase with pressure, but omits the flux constraints F_i^* which are constant; it also lists the corresponding values of β, I (total toroidal plasma current), and max ψ (poloidal flux at the magnetic axis). Figure 1 displays the magnetic surface plots corresponding to three of the instantaneous equilibria at $t = 0.0$, 0.5, 1.0. Figure 2 shows the normalized mid-line ($z = 0$) profiles of f, p, and J_ϕ for those equilibria.

TABLE I

Some Computed Quantities for the Slow Evolution Heating Problem

Time	G_1^*	G_2^*	G_3^*	G_4^*	G_5^*	β	I	max ψ
0.0	0.0513	0.0428	0.0309	0.0164	0.00284	0.0108	3.50	0.803
0.1	0.0613	0.0520	0.0382	0.0207	0.00365	0.0154	3.57	0.804
0.2	0.0718	0.0616	0.0461	0.0254	0.00452	0.0207	3.71	0.804
0.3	0.0823	0.0717	0.0543	0.0303	0.00546	0.0268	3.86	0.804
0.4	0.0942	0.0824	0.0630	0.0356	0.00648	0.0338	4.03	0.803
0.5	0.106	0.0936	0.0723	0.0413	0.00759	0.0415	4.22	0.803
0.6	0.119	0.105	0.0821	0.0473	0.00879	0.0508	4.43	0.803
0.7	0.132	0.118	0.0925	0.0539	0.0101	0.0597	4.65	0.803
0.8	0.146	0.131	0.104	0.0609	0.0115	0.0700	4.90	0.803
0.9	0.161	0.145	0.115	0.0685	0.0131	0.0812	5.17	0.802
1.0	0.177	0.160	0.128	0.0767	0.0147	0.0933	5.46	0.801

Finally, Table I also lists the corresponding values of β, I (total toroidal plasma current), and max ψ (poloidal flux at the magnetic axis). Figure 1 displays the magnetic surface plots corresponding to three of the instantaneous equilibria at $t = 0.0$, 0.5, 1.0. Figure 2 shows the normalized mid-line ($z = 0$) profiles of f, p, and J_ϕ for those equilibria.

This adiabatic evolution exhibits the behavior expected of flux-conserving plasma heating. Over the given time interval, β increases from 1.1% to 9.3% and correspondingly I increases from 3.5 to 5.5, while the magnetic surfaces shift outward and the toroidal current density profile becomes peaked on the outward side of the plasma. In these respects this family of solutions exhibits the expected behavior of flux-conserving tokamak equilibria [3, 4, 26]. For the purpose of comparison with other treatments, however, it is important to note that these results pertain to the relaxed problem with free-boundary formulated in Section 3, and as such are new.

The tabulation of max ψ in Table I demonstrates the degree to which flux is conserved by this family. While flux-conservation is imposed only in the volume-averaged sense that F_i ($i = 0, \ldots, 5$) are constant along the family, the variation of pointwise diagnostic max ψ is remarkably small. Similar results are observed over a wide range of conditions

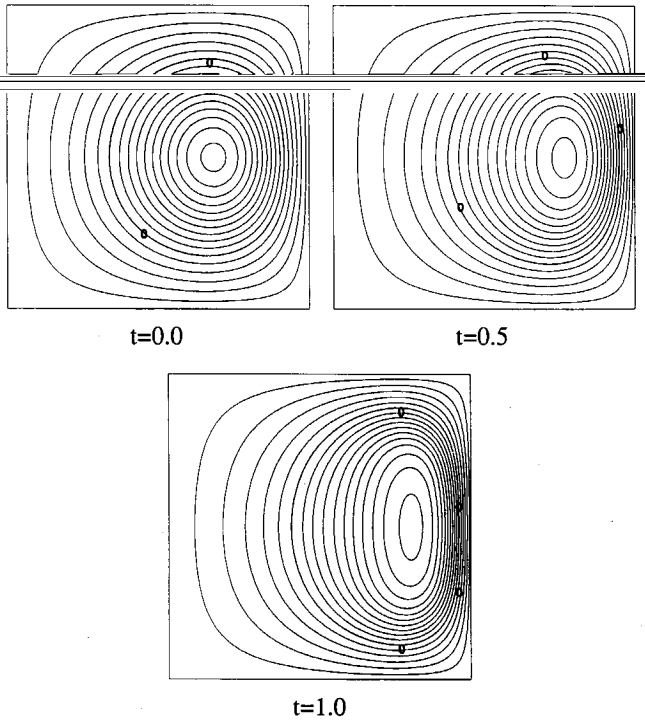

FIG. 1. Magnetic surface plots in poloidal cross section for the first family of equilibria; the plasma–vacuum interface is marked by some zeroes.

whenever (P_n) is solved with $n \geqslant 3$. On the other hand, $\max \psi$ can vary greatly when $n = 1$. This evidence demonstrates that exact (infinitely constrained) flux-conservation can be effectively approximated by a few integral constraints (say, $n = 5$). Since the computational effort of solving (P_n) grows rapidly with n, this property is very significant for practical implementations.

The irregularity of the current density profiles in Fig. 2 merits some comment since this effect is not commonly evidenced by simulations of flux-conserving equilibria under axisymmetry (which excludes the phenomena associated with rational magnetic surfaces and the formation of islands). The oscillatory behavior of these profiles is intrinsic to the solutions of the multiconstrained variational problem (P_n) and is observed over a wide range of conditions. It cannot be ascribed to spurious numerical error, since the spatial grid is sufficiently fine to resolve the solution of (P_n) accurately, provided that n is not large. It can, however, be taken as an indication of a lack of regularity of the solutions as n tends to infinity. We are led to suggest on this basis that the infinitely constrained problem (P_∞) has only *weak* solutions, and that as n tends to infinity the solutions of (P_n), which are smooth for each finite n, converge to a solution of (P_∞) only in a *weak* sense. Such a conjecture is consistent with the present lack of knowledge about the optimal regularity of weak solutions of the flux-conserving equilibrium problem and its prototypes [6, 8, 22, 23].

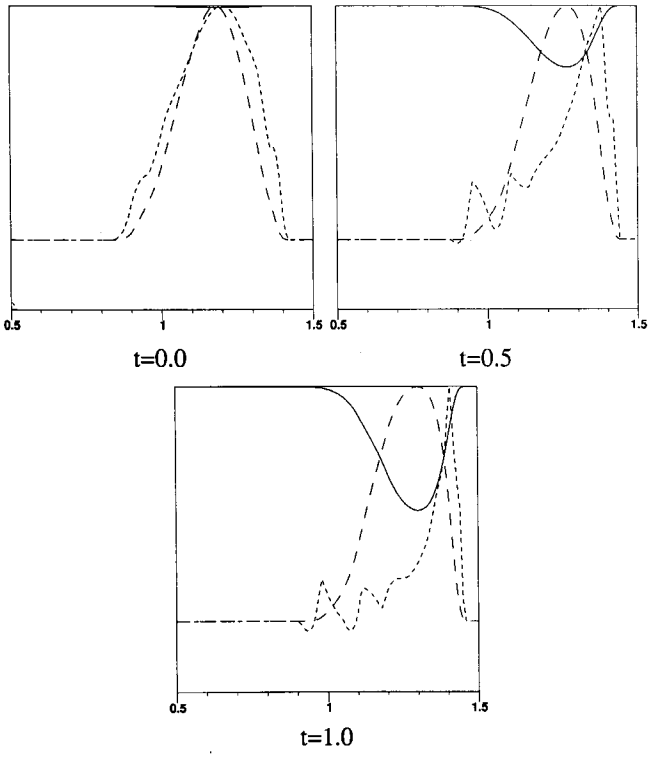

FIG. 2. Normalized profiles in the $z = 0$ mid-line for $n = 5$; f, p, and J_ϕ are indicated by bold, dashed, and dotted graphs, respectively.

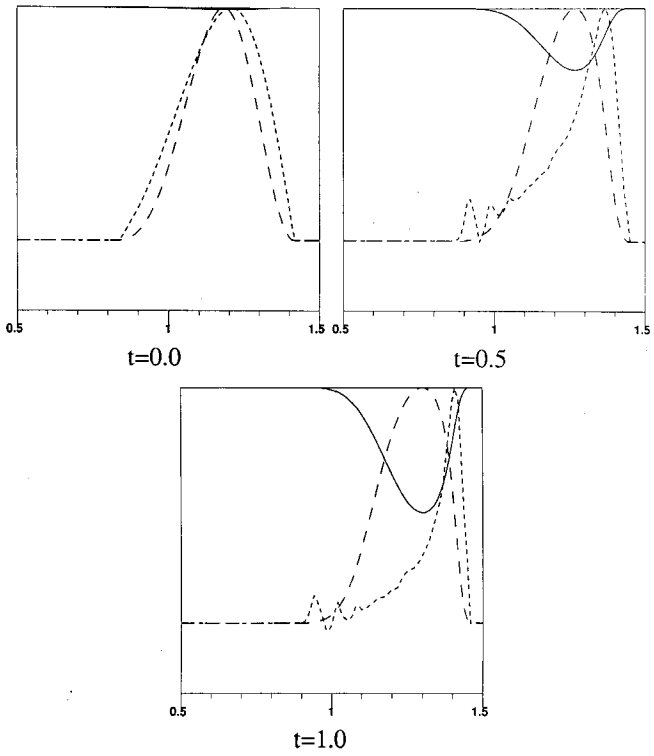

FIG. 3. Normalized profiles in the $z = 0$ mid-line for $n = 10$; f, p, and J_ϕ are indicated by bold, dashed, and dotted graphs, respectively.

Indeed, there is no known estimate (in an appropriate Sobolev norm) of the current density that is independent of n, apart from the obvious fact that the induced magnetic field has bounded energy. Consequently, the existence of singularities such as current sheets cannot be excluded at present. This possibility provides some further justification for the *relaxed* variational formulation that we adopt, in the sense that at least we are assured that the solutions we endeavor to compute have established analytical properties.

In order to examine the regularity question for (P_∞) (or its analogous GDE) we are compelled to consider the behavior of solutions of (P_n) as n is increased. To this end, we solve the same quasi-equilibrium problem as above, but with $n = 10$, to check quantitatively the effect of refining the discretization of the constraint families. In Fig. 3 we display in the normalized mid-line ($z = 0$) profiles corresponding to the solutions of (P_{10}) at the same times and for the same given data as those in Fig. 2. The current density profiles in this refined case continue to show some distinct irregularity (steep gradients and rapid oscillations), but the average behavior of the profiles appears to be more acceptable. These general features persist when the same problem is solved again with $n = 15$. Without being completely conclusive, these numerical tests support the conjecture that the regularity properties of infinitely constrained flux-conserving equilibria are delicate, while they also demonstrate that the average (large-scale) properties of the equilibria are

not sensitive to the (small-scale) irregularities. Moreover, these results appear to suggest that the singularities (if they exist) are located at or near the free-boundary and that the irregularities of the current profile in the plasma are related to singular behavior at the plasma–vacuum interface.

The second family of computed equilibria illustrates the change in shape of the magnetic surfaces during compression due to varying the external poloidal field. The computational domain $\Omega = \{2.5 < r < 3.5, |z| < 1.5\}$ is used with a grid having 101×101 nodes. The external field is induced by two elementary coils located at $(R_1, Z_1) = (2.7, 0)$ and $(R_2, Z_2) = (3.3, 0)$, carrying currents $I_1 = I_2 < 0$. The flux function $\bar{\psi}$ for this field is the sum of two Green functions for L corresponding to the coil locations (R_l, Z_l); their singular parts can be evaluated precisely by an explicit formula (in terms of elliptic integrals), and their regular parts can then be computed numerically. The problem is posed with $n = 3$ and equal flux increments $\Delta\sigma_i$ derived from a solution of (5.12), (5.13) with given parameters

$$a_0 = 5, \quad a_1 = -0.5, \quad a_2 = 5, \quad \kappa = 1, \quad I_0 = 1.5.$$

The constraint values fixed by this solution are

$$\begin{aligned} &F_0 = 3.184, \quad && \sigma_0 = 0.260 \\ &F_1 = 0.0502, \quad && F_2 = 0.0225, \quad && F_3 = 0.00356, \\ &G_1 = 0.0934, \quad && G_2 = 0.0543, \quad && G_3 = 0.00999. \end{aligned} \tag{6.2}$$

A family of six solutions of (P_3) is then generated by successively increasing the external coil currents $I_1 = I_2$, while maintaining all of these constraint values. We refer to the resulting solutions as equilibria (2.1)–(2.6). Table II records the prescribed values of I_1 and I_2, and the corresponding values of β, I, and max Ψ. Figure 4 shows the magnetic surface plots for these equilibria.

The most striking feature evidenced by this family is the large z-shift of the plasma from equilibrium (2.5) to (2.6), and the associated loss of z-symmetry. This behavior can be interpreted as the result of a *two-dimensional* (axisymmetric) instability that occurs at a critically compressed equilibrium, whereupon an energetically more favorable equilibrium is found by the (energy-decreasing) algorithm. By symmetry, the z-reflection of equilibrium (2.6) has identical properties, and so under these circumstances the algorithm may converge to either equilibrium (2.6) or its reflection, depending only on the small numerical errors that initiate a departure from z-symmetry. Consequently, many iterations may be needed to develop such a shift, if no other perturbations are added. It is noteworthy that max Ψ shows only a small variation in the course of this large change in the magnetic surfaces $\{\Psi = \sigma\}$, even though only three (volume-averaged) flux constraints are imposed.

It is possible to devise other geometries and external coil configurations which will exhibit bifurcation of the plasma region rather than the above z-shift. The variational formulation (P_n) remains unchanged through such a bifurcation, even though the interior of magnetic surfaces $\{\psi > \sigma\}$ may not be connected. (The integrals defining the constraints then extend over all of the components of these sets.) Thus, the above algorithm can be applied without any modifications to situations where the topology of the magnetic surfaces changes. In these situations our formula-

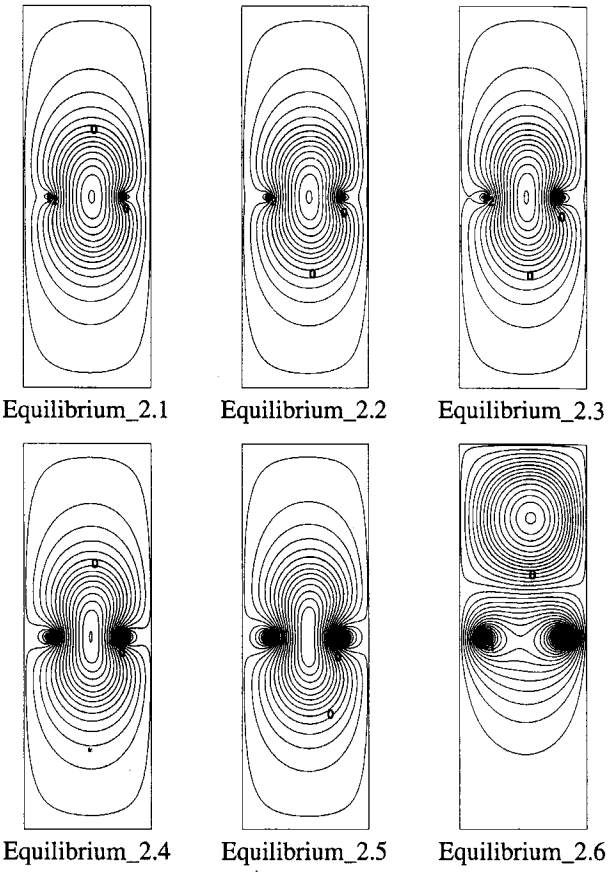

FIG. 4. Magnetic surface plots in poloidal cross section for the second family of equilibria; the plasma–vacuum interface is marked by some zeroes.

TABLE II

Some Computed Quantities for the Adiabatic Compression Problem

Equilibrium	I_1	I_2	β	I	max Ψ
(2.1)	−0.200	−0.200	0.0424	1.48	0.715
(2.2)	−0.260	−0.260	0.0423	1.56	0.715
(2.3)	−0.338	−0.338	0.0424	1.64	0.714
(2.4)	−0.439	−0.439	0.0423	1.76	0.712
(2.5)	−0.571	−0.571	0.0419	1.90	0.705
(2.6)	−0.742	−0.742	0.0437	1.51	0.718

tion and algorithm appear to have a distinct advantage over other methods, which encounter difficulties if the magnetic surfaces are not regularly nested.

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